Talk 1 – Shubham

Def^n A Ricci soliton structure is (M^n, g, X, λ) with (M^n, g) a Riemannian manifold, $X \in \Gamma(TM)$, $\lambda \in \mathbb{R}$ s.t.

$$Ric + \frac{1}{2}\mathcal{L}_X g = \frac{\lambda}{2} g \qquad \text{— } \text{RS}$$

$1/2$ factor is chosen just for computational ease.

Trace of RS gives

$$R + \operatorname{div} X = \frac{n\lambda}{2}.$$

When $X = \nabla f$ for some $f \in C^\infty(M)$ then we say that it is a gradient Ricci soliton (GRS) and we get

$$Ric + \operatorname{Hess} f = \frac{\lambda}{2} g \quad \text{or} \quad R_{ij} + \nabla_i \nabla_j f = \frac{\lambda}{2} g_{ij}.$$

→ $\mathbb{R}_+$ and $\mathrm{Diff}(M^n)$ act naturally as $\alpha \cdot g = \alpha g$ and $\varphi \cdot g = \varphi^* g$ on the space $\mathrm{Met}(M^n)$. ∵ Ric is scaling and diffeo invariant

$\Rightarrow$ $Ric(\alpha g) = Ric(g)$, $Ric(\varphi^* g) = \varphi^* Ric(g)$. $\therefore$ the action on a RS is

1) for $\alpha \in \mathbb{R}_+$ $(M^n, \alpha g, \alpha^{-1} X, \alpha^{-1}\lambda)$ is again a RS.

2) If $\varphi : N^n \to M^n$ is a diffeo, then $(N^n, \varphi^* g, \varphi^* X, \lambda)$ is a RS.

→ If K is a Killing v.f. then $(M^n, g, X+K, \lambda)$ is a RS.

Def^n :- A RS (M^n, g, X, λ) is shrinking if $\lambda > 0$ (shrinkers)

expanding if $\lambda < 0$ (expanders)

steady if $\lambda = 0$. (steadies)

→ We'll usually normalize so that λ is $1, -1$ or 0 in the above cases respectively.

Ricci solitons and the Ricci flow

Recall, if $g(t)$ is a RF on $M^n \times [a, b]$ then for any fixed $\alpha > 0$ and $\varphi \in Diff(M)$, $\tilde{g}(t) = \alpha(\varphi^* g)(t/\alpha)$ is again a RF on $M^n \times [\alpha a, \alpha b]$. So geometrically, these two sol^n are essentially the same.

Defn :- A sol^n $g(t)$ of RF which moves only by scalings and diffeos. is called a self-similar solution. i.e.

$g(t) = \alpha(t)\varphi_t^* g_0$ for some fixed g_0 and positive smooth function $c(t)$ and a family of diffeo φ_t.

A self-similar sol^n defines a RS on each time slice and conversely, a RS gives a self-similar sol^n to the RF.

Prop :- Let (M^n, g_0) be a Riem. manifold.

a) Suppose $g(t) = \alpha(t)\varphi_t^* g_0$ is a RF on $M^n \times (a,b)$ w/ $\alpha(t):(a,b)\to\mathbb{R}$ a positive smooth function, and φ_t a family of diffeos on $t\in(a,b)$.
Then $\forall$ $t\in(a,b)$ $\exists$ $X(t)\in\Gamma(TM)$ and a scalar $\lambda(t)$ s.t. $(M^n, g(t), X(t), \lambda(t))$ is a RS.

b) Suppose that (M^n, g_0, X, λ) is a RS. Then $\forall$ $x_0\in M$ $\exists$ a nbd $U\ni x_0$, an interval (a,b) containing 0, a smooth family $\varphi_t : U\to M^n$ of local diffeos and a smooth pos. function $\alpha:(a,b)\to\mathbb{R}$ s.t. $g(t)=\alpha(t)\varphi_t^* g_0$ is a RF on $U\times(a,b)$ w/ $g(0)=g_0$.

<u>Proof</u> :- Suppose $g(t) = \alpha(t)\varphi_t^* g_0$ is a RF on $M^n \times (a,b)$. Let $c \in (a,b)$. We have

$$\frac{\partial}{\partial t}\Big|_{t=c} g(t) = \alpha'(c)\varphi_c^* g_0 + \alpha(c)\frac{\partial}{\partial t}\Big|_{t=c} \varphi_t^* g_0$$

now $\frac{\partial}{\partial t}\Big|_{t=c} \varphi_t^* g_0 = \frac{\partial}{\partial t}\Big|_{t=0} (\varphi_c^{-1} \circ \varphi_{c+t})^* \varphi_c^* g_0 = \mathcal{L}_{X(c)} \varphi_c^* g_0$

where $X(c)$ is the generator of the family $\varphi_c^{-1} \circ \varphi_{c+t}$.

So, we get

$$\begin{aligned} -2Ric(g(c)) &= \alpha'(c)\varphi_c^* g_0 + \alpha(c)\mathcal{L}_{X(c)}\varphi_c^* g_0 \\ &= \varphi_c^*(\alpha'(c) g_0 + \alpha(c)\mathcal{L}_{X(c)} g_0) \end{aligned}$$

$$\Rightarrow \quad \varphi_c^*(\alpha'(c) g_0 + \alpha(c)\mathcal{L}_{X(c)} g_0 + 2Ric(g_0)) = 0.$$

$\therefore$ Choose $\lambda(c) = -\dfrac{\alpha'(c)}{\alpha(c)}$ gives a RS.

b) Suppose $Ric(g_0) + \frac{1}{2}\mathcal{L}_X g_0 = \frac{\lambda}{2} g_0$.

By the standard theory for existence of sol^n for ODE $\exists$ nbds $\mathcal{U}$ and $\mathcal{V}$ of $x_0 \in M$ w/ $\mathcal{U} \subset \mathcal{V}$, $\varepsilon > 0$ and a smooth family of diffeos s.t. $\psi_s : U \to V$, $s \in (-\varepsilon, \varepsilon)$ w/ $\psi_0(x) = x$ and

$$\frac{\partial}{\partial s}\Big|_{s=c} \psi_s(x) = X(\psi_c(x)). \quad \text{on } U \times (-\varepsilon, \varepsilon).$$

choose t s.t. $-\varepsilon < -\frac{1}{\lambda}\ln(1-\lambda t) < \varepsilon$ or $\frac{1-e^{-\lambda\varepsilon}}{\lambda} > t > \frac{1-e^{\lambda\varepsilon}}{\lambda}$

When $\lambda \neq 0$, define $b = \min\{\varepsilon, |\lambda|\}$ and $a = -b$ and $\forall t \in (a,b)$

let $\alpha(t) = 1 - \lambda t$, $\varphi_t = \psi_{c(t)}$ w/ $c(t) = -\frac{1}{\lambda}\ln(1-\lambda t)$.

Then $g(t) = \alpha(t)\varphi_t^* g_0 = (1-\lambda t)\varphi_t^* g_0$ satisfies $g(0) = g_0$

and
$$\frac{\partial}{\partial t} g(t) = \alpha'(t)\psi_{c(t)}^* g_0 + \alpha(t)\, c'(t) \cdot \psi_{c(t)}^* \mathcal{L}_X g_0$$

$$= -\lambda \varphi_t^* g_0 + \alpha(t)\left(\frac{1}{\alpha(t)}\right) \psi_{c(t)}^* \mathcal{L}_X g_0$$

$$= -\lambda \varphi_t^* g_0 + \varphi_t^*\left(-2\mathrm{Ric}(g_0) + \lambda g_0\right)$$

$$= -2\mathrm{Ric}(g(t)) \quad \text{on } U \times (a,b).$$

When $\lambda = 0$, choose $\alpha(t) = 1$ and $\varphi_t = \psi_t$. Then

$$\frac{\partial}{\partial t}\psi_t^* g_0 = \psi_t^* \mathcal{L}_X g_0 = -2\psi_t^* \mathrm{Ric}(g_0) = -2\mathrm{Ric}(g(t))$$

$\boxed{\text{M}}$

note :- If ω is some tensor on M then

$$\mathcal{L}_X \omega\big|_p = \frac{d}{dt}\Big|_{t=0} \varphi_t^* \omega\big|_p = \lim_{t\to 0} \frac{1}{t}\left(\varphi_t^*(\omega_{\varphi(p)}) - \omega_p\right)$$

w/ φ_t the associated flow of X.

$$\therefore \left.\frac{d}{dt}\right|_{t=t_0} \varphi_t^* \omega = \left.\frac{d}{dt}\right|_{t=0} \varphi_{t_0}^* \varphi_t^* \omega = \varphi_{t_0}^* \left.\frac{d}{dt}\right|_{t=0} \varphi_t^* \omega = \varphi_{t_0}^* \mathcal{L}_X \omega.$$

<u>Def^n</u> <u>complete vector field</u> $X \in \Gamma(TM)$ is a complete v.f. if it has a global flow, i.e. if $\forall\, p \in M$ and any integral curve $\gamma : (a,b) \to M$ passing through $p \in M$, can be extended to a curve $\gamma : \mathbb{R} \to M$.

$\therefore$ when X is a complete v.f., the domain of def^n in the proof of part b) would be at least as large as that permitted by the RS type, i.e. $\left(-\infty, \frac{1}{\lambda}\right)$ for shrinkers, $(-\infty, \infty)$ for steadies and $\left(-\frac{1}{\lambda}, \infty\right)$ for expanders.

<u>Some examples of Ricci solitons</u> :

① <u>The Gaussian soliton</u>

For $\lambda \in \mathbb{R}$, $(\mathbb{R}^n, g_{Euc.}, f_{Gau}, \lambda)$ w/ $g_{Euc} = \sum dx^i \otimes dx^i$ and $f_{Gau}(x) = \frac{\lambda}{4}|x|^2$ is called the Gaussian soliton.

$$\nabla_i f = \frac{\lambda}{2}\langle x, \nabla_i x\rangle, \quad \nabla_i\nabla_j f = \frac{\lambda}{2}\langle \nabla_j x, \nabla_i x\rangle + \frac{\lambda}{2}\langle x, \nabla_i\nabla_j x\rangle$$
$$= \frac{\lambda}{2} g_{ij}$$

$\therefore$ $\mathbb{R}^n$ can be regarded as a soliton of any type.

In fact, $f(x) = \frac{\lambda}{4}|x|^2 + \langle a, x\rangle + b$ w/ $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ also gives the same soliton structure.

note:- $\nabla f = \frac{\lambda}{2} x^i \frac{\partial}{\partial x^i}$ $\Rightarrow$ integrating it gives the 1-parameter family of diffeomorphisms $\tilde{\varphi}_t(x) = e^{\frac{\lambda}{2}t} x$

$\therefore$ from the previous prop., we see that

$$\varphi_t = \tilde{\varphi}_{-\frac{1}{\lambda}\ln(1-\lambda t)} \text{ when } \lambda \neq 0 \text{ and } \varphi_t = \tilde{\varphi}_t \text{ when } \lambda = 0$$

gives the family of diffeos, i.e.

$$\varphi_t(x) = (1-\lambda t)^{-1/2} x \quad \text{and} \quad g(t) = (1-\lambda t)\varphi_t^* g_{Euc}$$
$$= g_{Euc}. \text{ as expected.}$$

2. <u>Shrinking round Spheres</u>

$(\mathbb{S}^n, g_{round})$ is shrinking gradient Ricci solitons w/ $f \equiv$ constant. $g = 2(n-1) g_{S^n}$ satisfies. $Ric + \nabla^2 f = \frac{\lambda}{2} g$

w/ $\lambda = 1$. $\therefore$ we can take any constant function, let's choose $f = \frac{n}{2}$ and we get $(S^n, g, \frac{n}{2})$ as a shrinking gradient Ricci soliton.

The sol^n is $g(t) = (1-t)g$ which is defined for $t \in (-\infty, 1)$.

For $t < 1$, the metrics $g(t)$ have radius $r(t) = \sqrt{2(n-1)t}$.

3 Einstein manifolds

If (M^n, g, X, λ) is Einstein w/ $Ric = \frac{\lambda}{2} g$ then $\mathcal{L}_X g = 0$ and X is Killing.

4 Topping–Yin nongradient Ricci soliton

Consider $\mathbb{R}^2$ w/ $g = \frac{2}{1+y^2}(dx^2 + dy^2)$ and $X = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

Then $(\mathbb{R}^2, g, X, -1)$ is an expanding soliton.